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THE BOUNDARY VALUE PROBLEM OF ASYMMETRIC ELASTICITY THEORY IN QUASI-CLASSICAL APPROXIMATION

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Considered is the static boundary value problem of the asymmetric theory of elasticity for media in which the moment effects [couple stresses] supply a small contribution to the elastic energy. The elastic coefficients in the equilibrium equations, having the dimensions of a squared length, are taken to be small in comparison with the squares of the characteristic dimensions of the body. One obtains the solution of the equilibrium equations, containing small parameters in the leading derivatives. By an approximation method one constructs the solution for the field of displacements and rotations in the form of the sum of their classical limits and moment terms having the form of boundary layer functions. Boundary conditions of kinematic type are considered and a scheme is developed in order to satisfy them by the method of successive approximations.

Most of the media whose viscoelastic behavior is described within the limits of the asymmetric continuum mechanics (liquid) crystals, ferromagnetics, in a series of cases dislocation media and suspensions) are characterized by a small contribution of the moment terms in the general energetic balance of the deformation and flow processes. However, without taking into account the interaction of the moments one cannot give and interpretation to an entire series of delicate singularities of their viscoelastic behavior (the effects of the elastic distortion in the field of directions of the axes of molecular orientation in liquid crystals, the formation of spin waves in ferromagnetics, the effects of hardening in the plastic deformation, peculiarities of blood flow, etc.) In connection with this there arises the problem of analyzing the simplifications which can be introduced in the asymmetric mechanics by the investigation of media with weak moments.

Below we consider elastic isotropic media which are characterized by additional coefficients of rotational elasticity γ and moment elasticity η, τ, θ [1]. The energetic contribution of the moment terms to the elastic potential is determined by the ratio between these coefficients and the moduli λ, μ of the classical elasticity. If these ratios are small (in some sense which will be specified later), then we will say that the medium has weak moments [couple stresses]. For such media we investigate boundary value problem of the asymmetric elasticity theory.

1. Formulation of the problem. The equilibrium equations in the components of the displacement U and of the rotations Ω for an isotropic nongyrotropic medium can be written in the form [2]

$$(\lambda + 2\mu) \text{ grad div } \mathbf{U} - (\mu - \gamma) \text{ rot rot } \mathbf{U} - 2\gamma \text{ rot } \boldsymbol{\Omega} = 0 \qquad (1.1)$$
$$(\eta + \tau + \theta) \text{ grad div } \boldsymbol{\Omega} - \theta \text{ rot rot } \boldsymbol{\Omega} + 2\gamma\boldsymbol{\Omega} - \gamma \text{ rot } \mathbf{U} = 0$$

For the sake of brevity we do not consider body forces and body couples. We restrict ourselves to the analysis of boundary conditions of the kinematic type: on the boundary S of the simple connected domain V the displacements and rotations are given as functions of the coordinates q_2, q_3 of the boundary surface

$$\mathbf{U}\,|_s = \mathbf{V}\,(q_2, q_3), \qquad \boldsymbol{\Omega}\,|_s = \mathbf{G}\,(q_2, q_3) \qquad (1.2)$$

It is convenient to represent the system (1.1) in the reduced form [2]

$$(\lambda + 2\mu) \text{ grad div } \mathbf{U}^* - \mu \text{ rot rot } \mathbf{U}^* = 0 \qquad (1.3)$$

$$k_1^2 \text{grad div } \boldsymbol{\Omega}^* - k_2^2 \text{ rot rot } \boldsymbol{\Omega}^* - \boldsymbol{\Omega}^* = 0 \qquad (1.4)$$

Here

$$\mathbf{U} = \mathbf{U}^* - m^2 \operatorname{rot} \boldsymbol{\Omega}^*, \qquad \boldsymbol{\Omega} = \boldsymbol{\Omega}^* + {}^1/_2 \operatorname{rot} \mathbf{U}^*$$

$$k_1^2 = -(2\gamma)^{-1}(\eta + \tau + \theta), \qquad k_2^2 = -(2\mu\gamma)^{-1}\theta\,(\mu - \gamma), \quad m^2 = \theta / \mu \qquad (1.5)$$

The boundary conditions (1.2) for the new unknown functions $\mathbf{U}^*$, $\boldsymbol{\Omega}^*$ obtain the form

$$(\mathbf{U}^* - m^2 \operatorname{rot} \boldsymbol{\Omega}^*)\,|_S = \mathbf{V}, \qquad (\boldsymbol{\Omega}^* + {}^1/_2 \operatorname{rot} \mathbf{U}^*)\,|_S = \mathbf{G} \qquad (1.6)$$

We note that Eq. (1.3) coincides with the equilibrium equation of the classical elasticity theory.

The equilibrium equations of the asymmetric theory in the reduced form (1.3), (1.4) and the boundary conditions (1.6) contain three positive [2] coefficients k_1^2, k_2^2, m^2, representing the ratios of the elastic moduli μ, γ, η, τ, θ and having the dimension of a squared length. The real quantities k_1, k_2, m can be considered as characterizing the given medium by lengths. They serve as a measure of the deviation of the displacements and rotations from their classical limits and they allow to define qualitatively a medium with weak moments.

In order to introduce corresponding dimensionless parameters, we consider two linear measures which are characteristic for the given boundary value problem: the measure of the variation of the linear dimensions v° of the body (for example the displacement of some characteristic point of the body) and the space measure l. For the latter we can take a characteristic dimension of the body or of the domain of nonhomogeneity of the stress field. Then the relative quantities $\mathbf{U}^* / v^\circ$, $\boldsymbol{\Omega}^*$ are functions of the following dimensionless arguments: $x / l, y / l, z / l$, Γ, μ / λ, k_1^2 / l^2, k_2^2 / l^2, m^2 / lv°, where Γ is the collection of dimensionless parameters related to the geometry of the domain V

A medium with weak moments can be defined by the condition of smallness of the dimensionless parameters

$$k_1^2 / l^2, \quad k_2^2 / l^2, \qquad m^2 / lv^\circ \ll 1 \qquad (1.7)$$

Indeed, if these quantities are equal to zero, then it is obvious from (1.4), (1.5) that the displacement field is exactly equal to its classical limit and the rotations coincide with the curl of this field. However, it may happen, depending on the form of the function $\mathbf{G}$, that the second boundary condition (1.6) does not degenerate into the classical relation when the parameters (1.7) tend to zero. In addition, as it will be proved later, Eq. (1.4) has a solution which does not tend to zero at this limiting process. This degeneration of moment effects for residual phenomena at the boundary of the domain shall be called quasiclassical approximation in the asymmetric theory and it will be investigated in the sequel.

We begin the investigation of the boundary value problem (1.3), (1.4), (1.6) for small values of the parameters (1.7) with the analysis of Eq. (1.4) and then that of the boundary conditions (1.6). The solution of this equation can be reduced [2] to the solution of the Klein-Gordon scalar and vector equations

$$k_1^2\nabla^2\varphi - \varphi = 0, \qquad k_2^2\nabla^2\mathbf{\Phi} - \mathbf{\Phi} = 0 \tag{1.8}$$

and can be represented in the form

$$\Omega^* = \mathbf{\Phi} + k_1^2 \operatorname{grad}\varphi, \qquad \operatorname{div}\mathbf{\Phi} = 0 \tag{1.9}$$

2. The solution of the Klein-Gordon equation with a small parameter at the highest derivative. We consider the scalar equation (1.8), from which we will also obtain the solution of the vector equation. This equation contains a small parameter k_1 at the highest derivative and therefore its solution cannot be represented in the form of a power series in k_1 with uniform convergence in the entire domain including the boundary. The solution must be sought in the form of an asymptotic series in boundary layer type functions [3, 4].

It is convenient to represent the solution of the first equation of (1.8) in the form

$$\varphi = \chi^+ \exp(k_1^{-1}\ \Delta q) + \chi^- \exp(-k_1^{-1}\ \Delta q) \tag{2.1}$$

where Δq is the distance from the boundary to the given point measured in the direction of the exterior normal. The factors $\exp(k_1^{-1}\Delta q)$ and $\exp(-k_1^{-1}\Delta q)$ are standard boundary layer functions for the inner and outer domains, respectively. By definition, they are equal to unity on the boundary ($\Delta q = 0$) for arbitrary values of k_1, including zero. The functions $\chi^+, \chi^- = \chi(k_1, x, y, z)$ are determined from (1.8) by the method of successive approximations and usually have the form of asymptotic series in boundary layer functions. For their construction one selects a representation of the differential equation in a small neighborhood of the boundary which gives a convergent iteration process [3].

Below we suggest a representation of the first equation of (1.8) in the finite neighborhood of the boundary. This not only gives a convergent process of approximation but allows also to obtain χ^+, χ^- not as asymptotic but as power series in k_1. This representation is obtained in the so-called "layer" curvilinear coordinates q_1, q_2, q_3.

We construct in the neighborhood of the boundary a family of surfaces which are equidistant with respect to the surface S. On the latter we define an orthogonal system of surface curvilinear coordinates q_2, q_3 and we construct a coordinate line q_1, orthogonal to the family.

Obviously, the coordinate lines q_1 are straight lines. The Lamé coefficients are: $H_1 = 1$, $H_2 = H_2(q_1, q_2, q_3)$, $H_3 = H_3(q_1, q_2, q_3)$. In addition

$$\operatorname{grad}\Delta q = \mathbf{e}_1 \tag{2.2}$$

where $\mathbf{e}_1$ is the unit vector of the coordinate line q_1 having at the boundary the direction of the exterior normal. We denote the unit vectors of the coordinate lines q_2, q_3 by $\mathbf{e}_2, \mathbf{e}_3$.

We substitute (2.1) into the first equation of (1.8). Taking into account (2.2) we obtain a system of two conjugate equations for the functions χ^+, χ^-

$$k_1\nabla^2\chi^+ + \frac{2}{H}\frac{\partial H\chi^+}{\partial q_1} = 0, \quad k_1\nabla^2\chi^- - \frac{2}{H}\frac{\partial H\chi^-}{\partial q_1} = 0 \quad (H = \sqrt{H_2H_3}) \quad (2.3)$$

It is essential that the solutions of these equations are uniformly convergent with respect to k_1 in the entire domain including the boundary.

This follows from the fact that for $k_1 = 0$ there is no loss in boundary conditions for the system (2.3) as it happens in the case of the equations which give the boundary layer type solution [3]. Indeed, for $k_1 = 0$ from (2.3) we obtain a system of two equations of the first order, whose characteristics are normal to the boundary. Therefore χ^+, χ^- contain each an arbitrary function of the boundary coordinates q_2, q_3. With the help of these functions, as it will be proved in the sequel, one can satisfy all the boundary conditions of the Klein-Gordon equations.

The system (2.3) represents the first equation of (1.8) in a finite neighborhood of the boundary, called the "layer" zone. In general, the coordinates under consideration cannot be introduced in the entire domain irrespective of the geometry of its boundary, since the coordinate straight lines cannot intersect within the domain. The zone in which these intersections are missing (the layer zone), in a series of cases has a bounded width, which for a smooth convex surface cannot exceed the minimal positive radius of curvature of the surface. For nonsmooth surfaces one has to exclude the neighborhood of the edges.

In spite of the boundedness of the layer zone, the solution φ of the first equation of (1.8) can be extended to the entire domain taking into account the properties of the boundary layer type solution. This can be achieved in a particularly simple way in the case when the width of the boundary layer is substantially smaller than the width Q of the layer zone

$$k_1 / Q, \qquad k_2 / Q \ll 1 \qquad (2.4)$$

The asymptotic factors in (2.1), beyond the layer zone ($|\Delta q| > Q$) can be taken equal to zero. Then the relation (2.1) with the functions χ^+, χ^- obtained from (2.3), gives the solution of the first equation of (1.8) in the entire domain.

The condition (2.4) for media with weak moment is satisfied, as a rule, with a margin (for ferromagnetics $k_I \sim 10^{-5}$ cm). Otherwise, they must be considered as additional restrictions on the geometry of the boundary.

We consider now the determination of the functions χ^+, χ^-. We represent them, in agreement with the previous assertion, in the form of power series in k_1

$$\chi^\pm = \sum_{n=0}^{N} k_1{}^n \chi_n{}^\pm \qquad (2.5)$$

Substituting these series into (2.3), we obtain for $\chi_n{}^+$, $\chi_n{}^-$ a chain of equations

$$\frac{2}{H}\frac{\partial H\chi_n{}^\pm}{\partial q_1} = \mp \nabla^2\chi_{n-1}^\pm \qquad (n = 0, 1, 2, \ldots) \qquad (2.6)$$

(Here and in the sequel the superscripts correspond to the function χ^+, and the subscripts to the function χ^-). Solutions of (2.6) are given by the recursion relations

$$\chi_n{}^\pm = \frac{1}{2H}\left(a_n^\pm \mp \int H\nabla^2\chi_{n-1}^\pm dq_1\right) \qquad (\chi_{-1}^\pm = 0) \qquad (2.7)$$

where $a_n{}^+$, $a_n{}^-$ are arbitrary functions of the boundary coordinates q_2, q_3, which are determined from the boundary conditions.

For the sake of simplicity we consider Dirichlet boundary conditions

$$\varphi\,|_s = \varphi^s\,(q_2,\ q_3) \tag{2.8}$$

together with the conditions of boundedness at zero for the interior and at infinity for the exterior problem. Then from (2.1) we obtain for the interior problem

$$\chi^- = 0, \qquad \chi^+|_s = \varphi^s \qquad (\Delta q \leqslant 0) \tag{2.9}$$

and for the exterior problem

$$\chi^+ = 0, \qquad \chi^-|_s = \varphi^s \qquad (\Delta q \geqslant 0) \tag{2.10}$$

Satisfying these boundary conditions in each approximation with the help of (2.7), we obtain

$$\chi_0^+ = \chi_0^- = \frac{H^s}{H}\varphi^s, \quad \chi_{n+1}^+ = \chi_{n+1}^- = \frac{1}{2H}\int_0^{\Delta q} H\nabla^2\chi_n^{\pm}dq_1 \tag{2.11}$$

These expressions can be written in the form

$$\chi_n^+ = \chi_n^- = \frac{1}{H}R^n\varphi^s H^s, \quad R\varphi^s H^s = \frac{1}{2}\int_0^{\Delta q} H\nabla^2(H^{-1}\varphi^s H^s)\,dq_1 \tag{2.12}$$

$$(R^\circ\varphi^s H^s = \varphi^s H^s)$$

where R^n is the nth power of the operator R.

Substituting these formulas into (2.5) we obtain the functions χ^+ and χ^-

$$\chi^+ = \chi^- = \frac{1}{H}\left(\sum_{n=0}^{\infty} k_1{}^n R^n\right)\varphi^s H^s = \frac{1}{H}(1-k_1R)^{-1}\varphi^s H^s \quad (H^s = H\,|_s) \tag{2.13}$$

Since there is no explicit expression for the operator $(1 - k_1R)^{-1}$, one has to make use of its expansion in a series in k_1, restricting oneself to the necessary number of terms.

Let us verify the convergence of the approximation process under consideration. Making use of the formula for the finite sum of a decreasing geometric progression, we can write the remainder in (2.13)

$$g_{n+1} = \chi - H\,\frac{1-k_1{}^nR^n}{1-k_1R}\,\varphi^s H^s = \frac{Hk_1{}^nR^n}{1-k_1R}\,\varphi^s H^s$$

We form the remainder of the solution φ, i.e. $G_{n+1} = H^{-1}g_{n+1}\exp(k_1^{-1}\Delta q)$, and we substitute it into the first equation of (1.8). After transformation we obtain

$$(k_1{}^2\nabla^2 - 1)\,G_{n+1} = k_1^{n+1}\exp(k_1{}^{-1}\Delta q)\frac{\partial}{\partial q_1}(R^n\varphi^s H^s)$$

The operator R contains derivatives of the second order with respect to q_2, q_3 and derivatives of the first order with respect to q_1. If the boundary function φ^s and the geometric parameter H^s are $2n$ times differentiable with respect to q_2, q_3, and if H is $(n+1)$ times differentiable with respect to q_1, then the remainder in the equation is of order $k_1{}^{n+1}$.

Thus, the considered representation of the first equation of (1.8) in the layer zone allows us to obtain its approximate solution in a relatively simple form (2.1), (2.13). The equation is satisfied with the given degree of accuracy, while the boundary conditions are satisfied exactly.

The results obtained for the Klein-Gordon scalar equation can be extended to the case of the vector equation. We give the basic results

$$\mathbf{\Phi} = \mathbf{f}^{+} \exp(k_2^{-1}\Delta q) + \mathbf{f}^{-} \exp(-k_2^{-1}\Delta q) \tag{2.14}$$

$$\mathbf{f}^{\pm} = \sum_{n=0}^{N} k_2^{n}\mathbf{f}_n^{\pm}, \quad \mathbf{f}_n^{\pm} = \frac{1}{2H}\left(\mathbf{A}_n^{\pm} \mp \int H\nabla^2\mathbf{f}_{n-1}^{\pm} dq_1\right) \tag{2.15}$$

$$(\mathbf{A}_n^{\pm} = \mathbf{A}_n^{\pm}(q_2, q_3), \ \mathbf{f}_{-1}^{\pm} = 0, \ n = 0, 1, 2, \ldots)$$

3. The solution of the fundamental problem. We obtain the solution of the equation (1.4) for small k_1, k_2 by making use of the forms of the approximate solutions of the Klein-Gordon equations

$$\varphi = k_1^{-1}\chi \exp(k_1^{-1}\Delta q), \quad \mathbf{\Phi} = \mathbf{f} \exp(k_2^{-1}\Delta q) \ (\Delta q \leqslant 0) \tag{3.1}$$

We will consider only the interior problem, therefore we will omit the superscript at the quantities φ, χ, $\mathbf{\Phi}$, $\mathbf{f}$. In addition, in the first formula of (3.1) we have introduced for convenience the normalizing factor k_1^{-1}. As before, the amplitudes of the functions χ, f are determined by the formulas (2.5), (2.7), (2.15), but now the arbitrary functions a_n, $\mathbf{A}_n$ are determined from the boundary conditions (1.6).

We recall that Ω^* is expressed in terms of φ and $\mathbf{\Phi}$ with the formula (1.9) under the condition $\operatorname{div} \mathbf{\Phi} = 0$. We examine this condition. On the basis of (3.1) we have

$$\operatorname{div} \mathbf{\Phi} = (\operatorname{div} \mathbf{f} + k_2^{-1}f^{(1)}) \exp(k_2^{-1}\Delta q) \quad (f^{(1)} = \mathbf{e}_1\mathbf{f}) \tag{3.2}$$

The function $\operatorname{div} \mathbf{\Phi}$ is, obviously, a solution of the Klein-Gordon scalar equation. It is known [5] that if on the boundary of a simply connected domain a solution of this equation is equal to zero, then for $k_2^2 > 0$ it is equal to zero also in the interior. Therefore, in order to satisfy the condition $\operatorname{div} \mathbf{\Phi} = 0$ it is sufficient to set

$$(\operatorname{div} \mathbf{f})|_s = -k_2^{-1}f^{(1)}|_s \tag{3.3}$$

Representing here $\mathbf{f}$, $f^{(1)}$ in the form of series (2.15), we obtain for each approximation

$$f_n^{(1)}|_s = -(\operatorname{div} \mathbf{f}_{n-1})|_s \quad (\mathbf{f}_{-1} = 0, \ n = 0, 1, 2, \ldots) \tag{3.4}$$

These formulas represent the boundary conditions for $f_n^{(1)}$. Applying them to (2.15), we obtain

$$A_n^{(1)} = \left(\mathbf{e}_1 \int H\nabla^2\mathbf{f}_{n-1} dq_1\right)\Big|_s - 2H^s(\operatorname{div} \mathbf{f}_{n-1})|_s \ \ (A_n^{(1)} = \mathbf{e}_1\mathbf{A}_n) \tag{3.5}$$

Then from (2.15) we obtain

$$f_n^{(1)} = \frac{1}{2H}\left[\mathbf{e}_1 \int_{\Delta q}^{0} H\nabla^2\mathbf{f}_{n-1} dq_1 - 2H^s(\operatorname{div} \mathbf{f}_{n-1})|_s\right] \tag{3.6}$$

On the basis of (1.9), (3.1), (2.15), (2.5), the solution of Eq. (1.4) can be written in the form

$$\Omega^* = \exp(k_2^{-1}\Delta q) \sum_{n=0}^{N} k_2^{n}\mathbf{f}_n + \exp(k_1^{-1}\Delta q) \sum_{n=0}^{N} k_1^{n} (\mathbf{e}_1\chi_n + k_1 \operatorname{grad} \chi_n) \tag{3.7}$$

This solution contains in $\mathbf{f}_n$, χ_n three sequences of arbitrary functions $A_n^{(2)}$, $A_n^{(3)}$, a_n $(A_n^{(1)} = \mathbf{e}_i\mathbf{A}_n)$, which must be determined from the boundary conditions imposed on Ω^* and contained in (1.6). For a small parameter m the term $m^2 \operatorname{rot} \Omega^*$ in the first condition of (1.6) can be considered as a surface perturbation of the boundary condition $\mathbf{U}^*|_s = \mathbf{V}$. Then $\boldsymbol{\Omega}^*$ and $\mathbf{U}^*$ as solutions of linear equations can be represented in the form

$$\boldsymbol{\Omega}^* = \sum_{n=0}^{N} m^n \boldsymbol{\Omega}_n^*, \qquad \mathbf{U}^* = \sum_{n=0}^{N} m^n \mathbf{U}_n^* \tag{3.8}$$

The boundary conditions with respect to the functions $\boldsymbol{\Omega}_n^*$, $\mathbf{U}_n^*$ (but not with respect to $\boldsymbol{\Omega}^*$, $\mathbf{U}^*$!) can be separated. Practically, it is convenient to obtain at once the boundary conditions for $\mathbf{U}_n^*, \chi_n, \mathbf{f}_n$. To this end it is necessary to order (1.6) with respect to the small parameters k_1, k_2, m.

We rewrite the conditions (1.6), expressing $\boldsymbol{\Omega}^*$ from (3.7) and taking into account that on the boundary the exponential factors are equal to unity.

$$\mathbf{U}^*|_s - m^2 (k_2^{-1} [\mathbf{e}_1 \times \mathbf{f}] + \operatorname{rot} \mathbf{f})|_s = \mathbf{V}(q_2, q_3) \tag{3.9}$$

$$(\mathbf{f} + \mathbf{e}_1 \chi + k_1 \operatorname{grad} \chi + {}^1/_2 \operatorname{rot} \mathbf{U}^*)|_s = \mathbf{G}(q_2, q_3) \tag{3.10}$$

We consider the case when the parameters k_1, k_2, m have the same order of smallness and represent χ, $\mathbf{f}$ in the form of the series (2.5), (2.15) and $\mathbf{U}^*$ in the form of the series (3.8). Then (3.9) gives a chain of conditions on the boundary

$$\mathbf{U}_0^*|_s = \mathbf{V}, \quad \mathbf{U}_1^*|_s = \frac{m}{k_2} [\mathbf{e}_1 \times \mathbf{f}_0]|_s, \quad \mathbf{U}_{n+2}^*|_s = \left(\frac{k_2}{m}\right)^n ([\mathbf{e}_1 \times \mathbf{f}_{n+1}] + \operatorname{rot} \mathbf{f}_n)|_s \tag{3.11}$$

From (3.10) we also obtain a chain of conditions on the boundary

$$(\mathbf{f}_0 + \mathbf{e}_1 \chi_0)|_s = \mathbf{G} - {}^1/_2 (\operatorname{rot} \mathbf{U}_0^*)|_s \tag{3.12}$$

$$(k_2^{n+1} \mathbf{f}_{n+1} + \mathbf{e}_1 k_1^{n+1} \chi_{n+1})|_s = -(k_1^{n+1} \operatorname{grad} \chi_n + {}^1/_2 m^{n+1} \operatorname{rot} \mathbf{U}_{n+1}^*)|_s$$

The conditions (3.11), (3.12) allow to determine successively the boundary values of the functions $\mathbf{U}_n^*$, χ_n, $\mathbf{f}_n$. In order to separate the conditions (3.12) with respect to χ_n, $\mathbf{f}_n$, we project them onto the directions $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, denoting the projections by the superscript (1) - (3), respectively, and we take into account the boundary relations (3.4) for $f_n^{(1)}$. We obtain

$$\chi_0|_s = G^{(1)} - {}^1/_2 (\operatorname{rot}^{(1)} \mathbf{U}_0^*)|_s, \qquad f_0^{(2,3)}|_s = G^{(2,3)} - {}^1/_2 (\operatorname{rot}^{(2,3)} \mathbf{U}_0^*)|_s$$

$$k_1^{n+1} \chi_{n+1}|_s = \left(k_2^{n+1} \operatorname{div} \mathbf{f}_n - k_1^{n+1} \frac{\partial \chi_n}{\partial q_1} - {}^1/_2 m^{n+1} \operatorname{rot}^{(1)} \mathbf{U}_{n+1}^*\right)\Big|_s \tag{3.13}$$

$$k_2^{n+1} f_{n+1}^{(2,3)}|_s = \left(-\frac{k_1^{n+1}}{H_{2,3}} \frac{\partial \chi_n}{\partial q_{2,3}} - \frac{m^{n+1}}{2} \operatorname{rot}^{(2,3)} \mathbf{U}_{n+1}^*\right)\Big|_s$$

The relations (3.11), (3.13) represent recursion formulas which allow to compute successively the boundary values $\mathbf{U}_n^*|_s$, $\chi_n|_s$, $f_n|_s$. In this connection, in each approximation it is necessary to restore the functions $\mathbf{U}_n^*$, χ_n, $\mathbf{f}_n$ themselves which determine the boundary values for $\mathbf{U}_{n+1}^*$, χ_{n+1}, $\mathbf{f}_{n+1}$ in the next approximation. The scheme of the computation is the following:

$$\mathbf{U}_0^* \to (\chi_0, f_0^{(2,3)}) \to \mathbf{U}_1^* \to (\chi_1, f_1^{(2,3)}) \to$$

The functions $f_n^{(1)}$ are obtained separately from the relations (3.6).

We consider the first two approximations. In the zeroth approximation one finds the functions $\mathbf{U}_0^*$, χ_0, $\mathbf{f}_0$. The first function is obtained by solving the boundary value problem of the classical theory with the first boundary condition (3.11). From here we compute $(\operatorname{rot} \mathbf{U}_0^*)|_s$ and substitute it into (3.13) which gives the boundary values

$\chi_0|_s$, $f_0^{(2,3)}|_s$. Then, by the formulas (2.7), (2.15) the arbitrary functions a_0, $A_0^{(2,3)}$ are found together with the functions χ_0, $f_0^{(2,3)}$ themselves. We add to them $f_0^{(1)}$ from (3.6). As a result we obtain

$$a_0 = 2H^s [G_0^{(1)} - {}^1/_2 (\mathrm{rot}^{(1)}\mathbf{U}_0^*)|_s], \qquad A_0^{(2,3)} = 2H^s [G^{(2,3)} - {}^1/_2 (\mathrm{rot}^{(2,3)}\,\mathbf{U}_0^*)|_s]$$

$$\chi_0 = H^s H^{-1} [G^{(1)} - {}^1/_2 (\mathrm{rot}^{(1)}\mathbf{U}_0^*)|_s] \tag{3.14}$$

$$f_0^{(2,3)} = H^s H^{-1} [G^{(2,3)} - {}^1/_2 (\mathrm{rot}^{(2,3)}\,\mathbf{U}_0^*)|_s], \qquad f_0^{(1)} = 0$$

We consider now the first order approximation. Substituting the last two relations of (3.14) into the second condition (3.11) we obtain the boundary condition for $\mathbf{U}_1^*$. We solve once again the boundary value problem of the classical theory and on the basis of the obtained function $\mathbf{U}_1^*$ and then of the functions χ_0 and $\mathbf{f}_0$ the boundary values $\chi_1|_s$, $f_1^{(2,3)}|_s$ are computed from the second and the third relation (3.13).

As a result, by the method of successive approximations, one can satisfy the boundary conditions (1.6) with any degree of accuracy.

Thus, the method of construction of the solution of Eq. (1.4) with boundary conditions (1.6) reduces for small coefficients k_1, k_2, m to solving n times the equilibrium equation of the classical theory (1.3) with boundary conditions of kinematic type which gives the auxiliary functions $\mathbf{U}_n^*$ and to the subsequent determination of the functions a_n, $A_n^{(2)}$, $A_n^{(3)}$ from the boundary conditions, which with the aid of (2.7), (2.15) χ_n and $\mathbf{f}_n$ are obtained, which in turn give Ω^*

We return now to the boundary value problem of the asymmetric elasticity in the first formulation, i.e. equilibrium equations (1.1) with the boundary conditions (1.2). Its solution, i.e. the field of the displacements $\mathbf{U}$ and of rotations Ω is obtained in terms of the functions $\mathbf{U}_n^*$, χ_n, f_n, if in (1.5) one substitutes Ω^* from (3.7) and $\mathbf{U}^*$ from the second relation (3.8), in the form

$$\mathbf{U} = \sum_{n=0}^{N} m^n \mathbf{U}_n^* - \frac{m^2}{k_2} \exp(k_2^{-1}\Delta q) \sum_{n=0}^{N} k_2^n \, ([\mathbf{e}_1 \times \mathbf{f}_n] + k_2 \,\mathrm{rot}\,\mathbf{f}_n) \tag{3.15}$$

$$\Omega = \frac{1}{2} \sum_{n=0}^{N} m^n \,\mathrm{rot}\,\mathbf{U}_n^* + \exp(k_1^{-1}\Delta q) \sum^{N} k_1^n (\mathbf{e}_1\chi_n + k_1 \,\mathrm{grad}\,\chi_n) +$$

$$+ \exp(k_2^{-1}\Delta q) \sum_{n=0}^{N} k_2^n \mathbf{f}_n \quad (\Delta q \leqslant 0) \tag{3.16}$$

We would like to emphasize that the obtained solution is valid for small characteristic lengths k_1, k_2, m of the same order of smallness and for sufficiently shallow boundaries of the domain, i.e. under the conditions (1.7), (2.4).

Let us consider in detail the obtained solution. The first terms in each of the expressions (3.15), (3.16), i.e. $\mathbf{U}_0^*$ and ${}^1/_2\,\mathrm{rot}\,\mathbf{U}_0^*$ represent the "classical" components of the fields of the displacements $\mathbf{U}$ and the rotations Ω respectively. All the remaining terms are "moment" [couple] corrections. They have a double character.

One group of terms, containing $\mathbf{U}_1^*$, $\mathbf{U}_2^*$, . . ., determine the distortion of the field of displacements and rotations in the volume of the entire body. From the second boundary condition (3.11) and the second relation (3.13) it is clear that the source of the terms containing $\mathbf{U}_1^*$, are the rotations G at the boundary and ${}^1/_2\,\mathrm{rot}\,\mathbf{U}_0^*$ in the volume. The latter are related with the nonhomogeneity of the classical field of deformations. The term in (3.15) which contains U_1^* is of order $m^2 (k_2 v^\circ)^{-1}$, as it follows from the second relation (3.11) and the second relation (3.13). (Here by v° one can understand the largest displacement within the limits of the body.) This correction term may turn out to be essential, if v° is sufficiently small, which can occur for example in the field of ultrasonic waves.

The second group of terms, containing the exponential factors, give boundary effects of different orders with respect to k_1, k_2, m. They are significantly different from zero only for $|\Delta q| \lesssim k_1, k_2$, i.e. they are concentrated in the boundary layer of width $\sim k_1, k_2$ (from here we obtain a physical interpretation of these characteristic lengths). Near the boundary, as it follows from (3.14), the boundary layer terms of order zero give the relative torsion angle of $G - {}^1/_2 \operatorname{rot} U_0^*$, which may not be small in comparison with the angle of ${}^1/_2 \operatorname{rot} U_0^*$. Thus, the degeneration of a moment medium with respect to the elastic properties into a classical one, is accompanied by a loss of moment effects inside the body but not on its boundary.

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